PERTH MODERN SCHOOL
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**Semester Two Examination, 2020**

**Question/Answer booklet**

# MATHEMATICS SPECIALIST
# UNITS 1&2

## Section One:
## Calculator-free

Your Name________________________________

Your Teacher's Name______________________________

## Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

## Materials required/recommended for this section

***To be provided by the supervisor***
This Question/Answer booklet
Formula sheet

***To be provided by the candidate***
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

## Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

| Question | Mark | Max | Question | Mark | Max |
|---|---|---|---|---|---|
| 1 | | 4 | 6 | | 5 |
| 2 | | 9 | 7 | | 7 |
| 3 | | 7 | 8 | | 5 |
| 4 | | 9 | | | |
| 5 | | 4 | | | |

## Structure of this paper

| Section | Number of questions available | Number of questions to be answered | Working time (minutes) | Marks available | Percentage of examination |
|---|---|---|---|---|---|
| Section One: Calculator-free | 8 | 8 | 50 | 50 | 35 |
| Section Two: Calculator-assumed | 12 | 12 | 100 | 93 | 65 |
| | | | | **Total** | 100 |

## Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2020*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

**Section One: Calculator-free** **(50 marks)**

This section has **8 (eight)** questions. Answer **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 50 minutes.

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**Question 1 (2.3.1)** **(4 marks)**

Prove the following statement:

If $m$ and $n$ are odd integers, then $m^2 - n^2$ is divisible by 4.

| Solution |
|---|
| Assume that $m$ and $n$ are both odd integers.<br>Then $m = 2k + 1$ and $n = 2\ell + 1$ for some $k, \ell \in \mathbb{Z}$, and so<br>$m^2 - n^2 = (2k+1)^2 - (2\ell+1)^2$<br>$= 4k^2 + 4k + 1 - (4\ell^2 + 4\ell + 1)$<br>$= 4k^2 + 4k - 4\ell^2 - 4\ell$<br>$= 4(k^2 + k - \ell^2 - \ell)$<br>which is divisible by 4.<br>Hence $m^2 - n^2$ is divisible by 4.<br>QED |
| **Specific behaviours** |
| ✓ assumes $m$ and $n$ are odd<br>✓ expresses $m$ as $2k + 1$ and $n$ as $2\ell + 1$<br>✓expands and factorises $(2k+1)^2 - (2\ell+1)^2$<br>✓concludes that $m^2 - n^2$ is divisible by 4 |

**Question 2 (2.2.1 - 2.2.3, 2.2.11)** **(9 marks)**

a) Let $M = \begin{bmatrix} 3 & 4 \\ 1 & -2 \end{bmatrix}$.

i. Determine $M^{-1}$. (2 marks)

| Solution |
|---|
| $\det(M) = 3 \times (-2) - 4 \times 1 = -10$<br>Hence $M^{-1} = -\frac{1}{10}\begin{bmatrix} -2 & -4 \\ -1 & 3 \end{bmatrix}$ |
| **Specific behaviours** |
| ✓ calculates determinant<br>✓ writes correct expression for $M^{-1}$ |

ii. Showing use of an appropriate matrix equation together with your answer to part (i), determine the coordinates of the point of intersection of the lines $3x + 4y = -1$ and $x - 2y = 8$.

(3 marks)

| Solution |
|---|
| Lines intersect at $(x, y)$ where $\begin{bmatrix} x \\ y \end{bmatrix}$ satisfies:<br>$\begin{bmatrix} 3 & 4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$<br>Now<br>$M^{-1}\begin{bmatrix} 3 & 4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = M^{-1}\begin{bmatrix} -1 \\ 8 \end{bmatrix}$<br>$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{10}\begin{bmatrix} -2 & -4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 8 \end{bmatrix}$<br>$= -\frac{1}{10}\begin{bmatrix} -30 \\ 25 \end{bmatrix}$<br>Hence the lines intersect at $(3, -\frac{5}{2})$ |
| **Specific behaviours** |
| ✓ writes correct matrix equation<br>✓ multiplies both sides by $M^{-1}$<br>✓ states coordinates of point of intersection |

b) Consider the equation

$$\begin{bmatrix} 6 & 4 \\ k & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

where $k \in \mathbb{R}$.

i. Determine the value of $k$ such that the equation **does not** have a unique solution for $\begin{bmatrix} x \\ y \end{bmatrix}$. (2 marks)

| **Solution** |
|---|
| No unique solution if $\det \begin{bmatrix} 6 & 4 \\ k & -2 \end{bmatrix} = 0$, which is true if $-12 - 4k = 0$.<br>Hence if $k = -3$, the equation does not have a unique solution. |
| **Specific behaviours** |
| ✓ equates determinant to 0<br>✓ states $k = -3$ |

ii. With the value of $k$ obtained in part (i), what is the geometrical relationship between the lines $6x + 4y = -8$ and $kx - 2y = 4$? (2 marks)

| **Solution** |
|---|
| With $k = -3$, the first equation is $-2$ times the second. Hence the lines are coincident. |
| **Specific behaviours** |
| ✓ states that equations are scalar multiples of each other<br>✓ states that lines are coincident/same |

**Question 3 (2.3.4, 2.3.5)** **(7 marks)**

Use the principle of mathematical induction to prove that

$$4 + 32 + 108 + \cdots + 4n^3 = n^2(n+1)^2$$

for all integers $n \geq 1$.

| Solution |
|---|
| Let $P(n)$ stand for the statement '$4 + 32 + 108 + \cdots + 4n^3 = n^2(n+1)^2$' for all $n \in \mathbb{Z}^+$.<br><br>In $P(1)$:<br>$\text{LHS} = 4 \times 1^3 = 4$<br>$\text{RHS} = 1^2(1+1)^2 = 4$<br><br>Hence LHS = RHS and so $P(1)$ is true.<br><br>Now assume that $P(k)$ is true for some integer $k \geq 1$.<br>Then $4 + 32 + 108 + \cdots + 4k^3 = k^2(k+1)^2$.<br><br>Now LHS of $P(k+1) = 4 + 32 + 108 + \cdots + 4k^3 + 4(k+1)^3$<br>$= k^2(k+1)^2 + 4(k+1)^3$<br>$= (k+1)^2(k^2 + 4(k+1))$<br>$= (k+1)^2(k^2 + 4k + 4)$<br>$= (k+1)^2(k+2)^2$<br>$= (k+1)^2((k+1)+1)^2$<br>$=$ RHS of $P(k+1)$<br><br>This shows that $P(k+1)$ is also true.<br><br>Hence, by PMI, $P(n)$ is true for all integers $n \geq 1$. |
| **Specific behaviours** |
| ✓ defines $P(n)$<br>✓ shows that $P(1)$ is true by evaluating LHS and RHS separately<br>✓ assumes $P(k)$ is true<br>✓ writes LHS of $P(k+1)$ in terms of RHS of $P(k)$<br>✓ simplifies expression to obtain $(k+1)^2(k+2)^2$<br>✓ concludes that $P(k+1)$ is also true<br>✓ concludes proof by referring to PMI |

**Question 4 (2.3.7-2.3.10)** **(9 marks)**

Let $z = 3 - 5i$ and $w = -2 + i$. Write each of the following in the form $a + bi$ where $a, b \in \mathbb{R}$.

a) $z + w$ (2 marks)

| **Solution** |
|---|
| $z + w = 3 - 2 - 5i + i$<br>$= 1 - 4i$ |
| **Specific behaviours** |
| ✓ adds real and imaginary parts separately<br>✓ states $1 - 4i$ |

b) $3zw$ (2 marks)

| **Solution** |
|---|
| $3zw = 3(3 - 5i)(-2 + i)$<br>$= 3(-6 + 3i + 10i - 5i^2)$<br>$= -3 + 39i$ |
| **Specific behaviours** |
| ✓ expands correctly<br>✓ states $-3 + 39i$ |

c) $z + \overline{z}$ (2 marks)

| **Solution** |
|---|
| $z + \overline{z} = 3 - 5i + 3 + 5i$<br>$= 6$ |
| **Specific behaviours** |
| ✓ determines conjugate of $z$ or uses $z + \overline{z} = 2\text{Re}(z)$<br>✓ states $6$ |

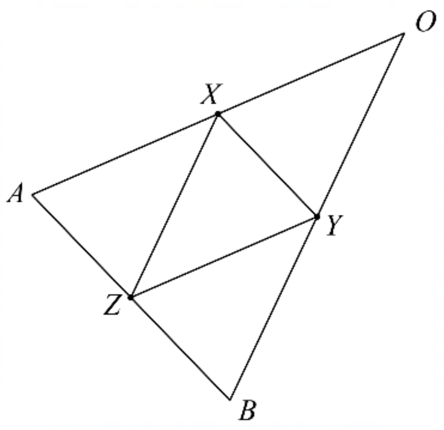
d) $\frac{z}{w}$ (3 marks)

| **Solution** |
|---|
| $\frac{z}{w} = \frac{3 - 5i}{-2 + i} \times \frac{-2 - i}{-2 - i}$<br>$= \frac{(3 - 5i)(-2 - i)}{5}$<br>$= \frac{-11 + 7i}{5}$<br>$= -\frac{11}{5} + \frac{7}{5}i$ |
| **Specific behaviours** |
| ✓ multiplies numerator and denominator by $\overline{w}$<br>✓ expands<br>✓ states $-\frac{11}{5} + \frac{7}{5}i$ (must be in form $a + bi$ i.e. don't accept $\frac{-11+7i}{5}$ as final answer) |

**Question 5 (1.2.11, 1.2.12)** **(4 marks)**

Let $\Delta OAB$ be an isosceles triangle with $OA = OB$, and let $X, Y$ and $Z$ be the midpoints of $\overline{OA}$, $\overline{OB}$ and $\overline{AB}$ respectively. Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$.

Use a vector method to prove that $\Delta XYZ$ is isosceles.




| Solution |
|---|
| $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$<br>$\overrightarrow{XZ} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$<br>$= \frac{1}{2}\boldsymbol{a} + \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$<br>$= \frac{1}{2}\boldsymbol{b}$<br>$\overrightarrow{YZ} = \frac{1}{2}\overrightarrow{OB} - \frac{1}{2}\overrightarrow{AB}$<br>$= \frac{1}{2}\boldsymbol{b} - \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$<br>$= \frac{1}{2}\boldsymbol{a}$<br>Thus $\lvert\overrightarrow{XZ}\rvert = \frac{1}{2}\lvert\boldsymbol{b}\rvert$ and $\lvert\overrightarrow{YZ}\rvert = \frac{1}{2}\lvert\boldsymbol{a}\rvert$.<br>But since $\Delta OAB$ is isosceles, $\lvert\boldsymbol{a}\rvert = \lvert\boldsymbol{b}\rvert$, and hence $\lvert\overrightarrow{XZ}\rvert = \lvert\overrightarrow{YZ}\rvert$, meaning that $\Delta XYZ$ is isosceles. |
| **Specific behaviours** |
| ✓ writes $\overrightarrow{AB}$ as $\boldsymbol{b} - \boldsymbol{a}$<br>✓ shows that $\overrightarrow{XZ} = \frac{1}{2}\boldsymbol{b}$<br>✓ shows that $\overrightarrow{YZ} = \frac{1}{2}\boldsymbol{a}$<br>✓ deduces from $\lvert\boldsymbol{a}\rvert = \lvert\boldsymbol{b}\rvert$ that $\lvert\overrightarrow{XZ}\rvert = \lvert\overrightarrow{YZ}\rvert$ |

**Question 6 (2.1.3 - 2.1.6, 2.1.8)** **(5 marks)**

Prove that the following is true for all $\theta$.

$$2\sin\theta - \sin 2\theta \cos\theta = 2\sin^3\theta$$

| **Solution** |
|---|
| $\text{LHS} = 2\sin\theta - \sin 2\theta\cos\theta$<br>$= 2\sin\theta - 2\sin\theta\cos\theta\cos\theta$<br>$= 2\sin\theta - 2\sin\theta\cos^2\theta$<br>$= 2\sin\theta\,(1-\cos^2\theta)$<br>$= 2\sin\theta\sin^2\theta$<br>$= 2\sin^3\theta$<br>$= \text{RHS}$<br>Hence $2\sin\theta - \sin 2\theta\cos\theta = 2\sin^3\theta$. |
| **Specific behaviours** |
| ✓ expands $\sin 2\theta$ using double angle formula<br>✓ factorises out $2\sin\theta$<br>✓ using Pythagorean identity<br>✓ simplifies to $2\sin^3\theta$<br>✓ works from LHS to RHS (or vice versa) |

**Question 7 (1.3.2, 1.3.5, 2.3.1)** **(7 marks)**

a) Let $p$ be an irrational number and $q$ a rational number. Use the method of proof by contradiction to prove that $pq$ is irrational. (4 marks)

| **Solution** |
|---|
| Assume that $p$ is irrational and $q$ is rational, but that $pq$ is rational.<br>Then $q = \frac{a}{b}$ and $pq = \frac{c}{d}$ for some integers $a$, $b$, $c$ and $d$.<br>Now<br>$$\begin{aligned} p &= pq \div q \\ &= \frac{c}{d} \div \frac{a}{b} \\ &= \frac{c}{d} \times \frac{b}{a} \\ &= \frac{cb}{da} \end{aligned}$$<br>which is rational since $cb$ and $da$ are integers. This contradicts the assumption that $p$ is irrational; hence $pq$ must be irrational. |
| **Specific behaviours** |
| ✓ assumes that $pq$ is rational<br>✓ writes $q$ and $pq$ as a ratio of integers<br>✓ shows that $\frac{pq}{q}$ is rational<br>✓ notes contradiction and concludes that $pq$ is irrational |

b) State whether the following is true or false and prove or disprove it accordingly:

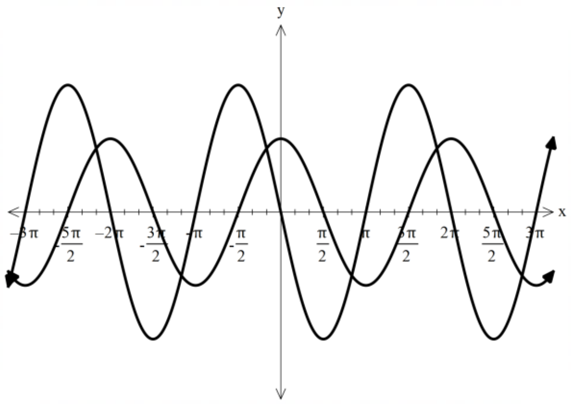
'If $p$ is irrational and $q$ is rational, then $(p+q)^2$ is irrational.'

(3 marks)

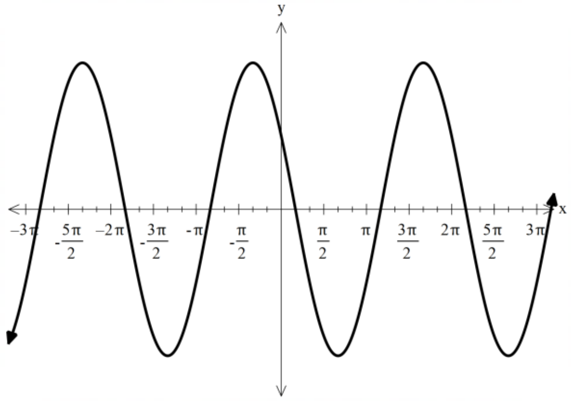
| **Solution** |
|---|
| The statement is false.<br>E.g. let $p = \sqrt{2} - 1$ and $q = 1$. Then $p$ is irrational and $q$ is rational, but $(p+q)^2 = 2$, which is rational. |
| **Specific behaviours** |
| ✓ states false<br>✓ gives counterexample values for $p$ and $q$<br>✓ shows that $(p+q)^2$ is rational for those values |

**Question 8 (2.1.7)** **(5 marks)**

The diagram below shows the graphs of two functions $f(x) = 3\cos(x)$ and $g(x) = a\sin(x)$ where $a$ is a real constant.




Below is the graph of $y = f(x) + g(x)$, which can also be expressed as $y = b\sin(x + \alpha)$ where $b$ and $\alpha$ are positive real constants.




a) Determine an exact value for $\alpha$. (1 mark)

| Solution |
|---|
| $\alpha = \frac{5\pi}{6}$ |
| **Specific behaviours** |
| ✓ states correct value |

b) Determine the values of $a$ and $b$. (4 marks)

| Solution |
|---|
| Equation of graph is $$y = b\sin\left(x + \frac{5\pi}{6}\right)$$ $$= b\sin x\cos\frac{5\pi}{6} + b\cos x\sin\frac{5\pi}{6}$$ $$= b\left(-\frac{\sqrt{3}}{2}\right)\sin x + b\frac{1}{2}\cos x$$ Since the equation of the graph is also $y = a\sin x + 3\cos x$, we have $\frac{b}{2} = 3$, so $b = 6$. Hence $a = 6\left(-\frac{\sqrt{3}}{2}\right) = -3\sqrt{3}$. |
| **Specific behaviours** |
| ✓ notes that $y = b\sin\left(x + \frac{5\pi}{6}\right)$<br>✓ expands using compound angle formula |

✓ solves for $b$ and states correct value
✓ solves for $a$ and states correct value

**END OF SECTION ONE**

**Additional working space**

Question number:

**Additional working space**

Question number: